Analysis questions 24/7 Evening

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.
(a) Show that if f is continuous, then it is linear.
(b) Show that if f is Borel measurable, then it is continuous and thus linear.

2. Suppose that μ is a Borel measure of total mass 1, and let $\hat{\mu}(\xi) = \int e^{-ix\xi}\mu(dx)$ be its Fourier transform. Show that if the $\hat{\mu}(\xi_0) = 1$ for some ξ_0, then the support of μ is contained in a translation and dilation of $\mathbf{Z}$, i.e., it is contained in $\{ak + b : k \in \mathbf{Z}\}$ for some real a and b.